A model of supersonic flow past blunt axisymmetric bodies, with application to Chester's solution

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SUMMARY

A simple approximate model is set forth for the flow field between the nose of a blunt body of revolution and its detached shock wave. The model tends to explain the poor convergence of Chester's solution, which is based on an improvement of the Newtonian approximation. It suggests a modification of his series for the body shape which appears to improve its convergence considerably.

1. INTRODUCTION

Chester (1956) and Freeman (1956) have recently advanced in this journal an ingenious attack on the problem of supersonic flow past a blunt body. (See also Lighthill 1957, §3.5 *et seq.*) Their point of departure is the 'Newtonian-plus-centrifugal' solution, which becomes exact as the free-stream Mach number M approaches infinity and the adiabatic index γ approaches unity. After modifying this basic solution to be valid near the body, Chester and Freeman improve upon it by successive approximations, the key to their success being the adoption of von Mises' transformation, in which the stream function is taken as one of the independent variables. The result of Chester (who has carried the approximation several steps further than Freeman for a perfect gas with constant specific heats) is a double series expansion in $\delta = (\gamma-1)/(\gamma+1)$ and M^{-2}.

Unfortunately, the series for the flow variables of interest appear to converge poorly and, surprisingly, less well for axisymmetric than plane flow. For example, for axisymmetric flow of a perfect gas with $\gamma = 7/5$ at $M = \infty$ past whatever body it is that supports a parabolic detached shock wave, Chester's series for the stand-off distance Δ of the shock wave, in terms of its nose radius R_s, is

$$\Delta/R_s = \tfrac{1}{6}(1-0{\cdot}6667+0{\cdot}4333-0{\cdot}3062+\ldots). \tag{1}$$

Although the expansion parameter δ is here only $1/6$, this series converges so slowly as to be of little practical value. (Chester has used his series only out to $\delta = 1/10$, and then attempted to extrapolate to such 'large' values as $1/6$.)

The question naturally arises whether this discouraging behaviour is symptomatic of some basic defect in the approximation, or simply indicates that the series has an unduly small radius of convergence (if it in fact converges). The present note aims to suggest that the latter is the case, and that by appeal to a simple approximate model of the flow field the series for the shape of the body can be recast in a form that appears to converge much more rapidly.

2. A simple model of the flow field

Consider axisymmetric flow of a perfect gas with constant γ behind a prescribed shock wave. For simplicity we take the shock wave to be a parabola (which is the only case considered in detail by Chester), but other shapes could be treated similarly. According to the Cauchy–Kowalewski theorem, the flow field is analytic in some region downstream of the shock wave, and there is no reason to doubt that this region extends to the surface of the body. One can therefore attempt to expand the flow field in Taylor series starting from the shock wave. It is convenient to work with Stokes's stream function ψ and parabolic coordinates (ξ, η) which are normalized such that the shock wave is described by $\eta = 1$.

On the downstream face of the shock wave, the stream function has the free-stream value, its first derivatives are given by the Rankine–Hugoniot shock relations, and its second derivatives can be found by substituting into the equations of motion. The present model consists of the resulting first three terms of the Taylor expansion starting from the shock wave:

$$\frac{\psi}{\xi^2} = \frac{1}{2} - \frac{(\gamma+1)M^2}{A}(1-\eta) + \left[2\left(\frac{M^2-1}{A}\right)^2 + \frac{(\gamma-1)M^2-2(2+M^2)}{2A} + \right.$$
$$+2\frac{M^2+1}{A^2}\xi^2 + \frac{4(\gamma+1)M^4\xi^2}{(M^2-1-\xi^2)A^2} + 2(\gamma-1)M^4\xi^2 \times$$
$$\left. \times \frac{2\gamma M^2-(\gamma-1)(1+\xi^2)}{(1+\xi^2)A^3}\right](1-\eta)^2, \quad (2\,\text{a})$$

where
$$A = 2(1+\xi^2)+(\gamma-1)M^2. \quad (2\,\text{b})$$

Even this approximation is rather unwieldy, but at $M = \infty$ it simplifies to

$$\frac{\psi}{\xi^2} = \frac{1}{2} - \frac{\gamma+1}{\gamma-1}(1-\eta) + \left[\frac{3-\gamma}{(\gamma-1)^2} + \frac{1}{2} + \frac{4\gamma}{(\gamma-1)^2}\frac{\xi^2}{1+\xi^2}\right](1-\eta)^2. \quad (3)$$

In the Newtonian limit, $\gamma \to 1$ (and $\eta \to 1$), this reduces further to

$$\frac{\psi}{\xi^2} \sim \frac{1}{2}\left(1-2\frac{1-\eta}{\gamma-1}\right)^2 + 4\frac{\xi^2}{1+\xi^2}\left(\frac{1-\eta}{\gamma-1}\right)^2. \quad (4)$$

On the axis ($\xi = 0$) the point at which this last expression vanishes (corresponding to the nose of the body) lies at $(1-\eta) = \frac{1}{2}(\gamma-1)$, which is just the result of the first approximation of Chester and Freeman. It is mainly this attribute that qualifies the model as a useful one. Off the axis (4) cannot vanish, which suggests that the model will prove less faithful away from the nose of the body.

It should be remarked that the idea of expanding the flow field in Taylor series from an assumed shock wave is by no means new, but has been followed by a number of workers with uniformly poor results. The innovations in the present model are, first, that the natural coordinates associated with the shock wave are employed, and, second, that the Taylor series is terminated with the parabolic approximation to the stream function; and both of these are essential. Previous investigators, of whom the most assiduous has been Cabannes (1951, 1956), all carried out double Taylor series expansions in Cartesian coordinates. Aside from the inferiority of a double series to the present single expansion, this yields series whose first few terms approach the true result much more slowly than the present series in parabolic coordinates. This is illustrated in figure 1 for a parabolic shock wave at $M = 2$ (with $\gamma = 7/5$), for which Cabannes (1956) has computed seven terms of the double Taylor series.

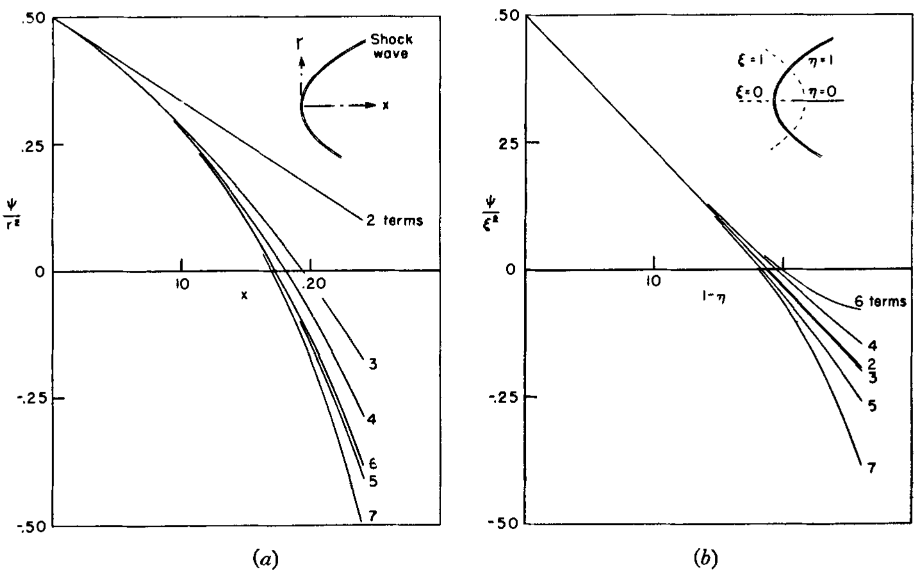

Figure 1. Taylor series expansions for stream function on axis of symmetry behind parabolic shock wave at $M = 2, \gamma = 7/5$. (*a*) Cartesian coordinates; (*b*) Parabolic coordinates.

Figure 1 (*b*) also illustrates the second point that, even in the natural coordinates, adding further terms to the Taylor series worsens rather than improves the approximation. The reason for this is that analytical continuation of the flow field upstream through the detached shock wave leads to a limiting line, or envelope of characteristics, as sketched in figure 2. The continued flow field has a square-root behaviour there. Since the Taylor series represents this analytical continuation, the limiting line determines its radius of convergence. On the axis, the limiting line is about as far ahead of the shock wave as the body is behind it, so that the convergence is marginal; and in the example of figure 1 it is actually closer, so that the series diverges at the body. (The divergence can be ameliorated by applying the transformation of Shanks (1955).) Off the axis the limiting line is even closer, relative to the distance of the body, so that the convergence deteriorates further.

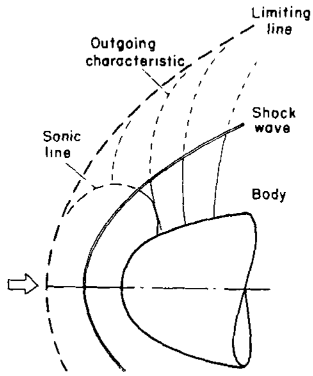

Figure 2. Schematic diagram showing analytical continuation of flow upstream through shock wave.

Poor convergence, or even divergence, is not a fatal flaw, however; though it is for this reason that the truncated series is regarded as a model rather than a systematic approximation. The variation of the stream function is actually very nearly parabolic, so that (somewhat as with asymptotic series) the first three terms of the Taylor series provide a close approximation near the axis.

3. Application to Chester's solution

3.1. *Body shape at infinite Mach number*

Consider first the case of infinite Mach number. Requiring the model stream function (3) to vanish determines the body according to

$$(1-\eta)_{\text{model}} = \frac{\delta}{1+[3\delta(1-\delta)-2(1-\delta^2)\xi^2/(1+\xi^2)]^{1/2}}, \tag{5}$$

where $\delta = (\gamma-1)/(\gamma+1)$. For comparison with Chester's solution, we transform to cylindrical polar coordinates in which the shock wave is described by $x = \frac{1}{2}r^2$, which means that $x = \frac{1}{2}(1+\xi^2-\eta^2)$ and $r = \xi\eta$, and then expand for small δ. Then (with fractions unreduced to facilitate comparison) the body is described by

$$\begin{aligned} x_{\text{model}} = \delta\left[1-\sqrt{(9\delta/3)}+\frac{25}{10}\delta-\frac{252}{168}\delta\sqrt{(9\delta/3)}+\ldots\right]+ \\ +\frac{1}{2}r^2\left[1+\frac{2}{3}\sqrt{(9\delta/3)}-2\delta+5\delta\sqrt{(9\delta/3)}-\right. \\ \left.-23\delta^2+\frac{343}{12}\delta^2\sqrt{(9\delta/3)}+\ldots\right], \end{aligned} \tag{6}$$

whereas Chester's result, of which the leading term was first given by Hayes (1955), is

$$x = \delta\left[1-\sqrt{(8\delta/3)}+\frac{26}{10}\delta-\frac{463}{168}\delta\sqrt{(8\delta/3)}+\ldots\right]+ \\ +\frac{1}{2}r^2\left[1+\frac{19}{6}\delta-\frac{47}{15}\delta\sqrt{(8\delta/3)}+\frac{11173}{840}\delta^2- \\ -\frac{2593}{126}\delta^2\sqrt{(8\delta/3)}+\ldots\right]. \quad (7)$$

The agreement is remarkable for the terms independent of r, though (as anticipated) less satisfactory for those in r^2. The model series converges for δ smaller than 3/9, which suggests that Chester's series will converge for δ less than 3/8, corresponding to γ less than 11/5, a value not attained in reality. However, the convergence of both model and prototype is unsatisfactory at $\delta = 1/6$, corresponding to $\gamma = 7/5$. In the model the poor convergence results from expansion of $[1+(3\delta)^{1/2}]^{-1}$ by the binomial theorem, which suggests that the convergence of Chester's series might be considerably accelerated by recasting it as a series for $(1-\eta)^{-1}$. Doing so yields on the axis ($\xi = 0$)

$$(1-\eta)^{-1} = \delta^{-1}\left[1+\sqrt{(8\delta/3)}-\frac{13}{30}\delta+\frac{187}{840}\delta\sqrt{(8\delta/3)}+\ldots\right]. \quad (8\,\text{a})$$

For $\delta = 1/6$ this gives

$$(1-\eta)^{-1} = 6(1+0{\cdot}6667-0{\cdot}0722+0{\cdot}0247+\ldots), \quad (8\,\text{b})$$

which appears to be a great improvement over the result of the original series for the related quantity Δ/R_s given in equation (1).

3.2. *Body shape at high Mach number*

Consider now Chester's general case of high Mach number. For simplicity, we restrict the model to the axis, where it is most faithful. Then from (2a) it locates the nose of the body at

$$(1-\eta)_{\text{model}} = [2+(\gamma-1)M^2][(\gamma+1)M^2+\{6(\gamma-1)M^4+ \\ +2(\gamma+5)M^2+4\}^{1/2}]^{-1}. \quad (9\,\text{a})$$

In terms of Chester's parameters

$$\delta = (\gamma-1)/(\gamma+1), \qquad d = \delta+M^{-2},$$

this is

$$(1-\eta)_{\text{model}} = (\delta+M^{-2}-M^{-2}\delta)\{1+[(1-\delta)(3\delta+4M^{-4}-3M^{-4}\delta)]^{1/2}\}^{-1} \quad (9\,\text{b})$$

or

$$(1-\eta)_{\text{model}} = d(1-M^{-2}+M^{-4}/d)\{1+[(1-d+M^{-2})(3d-3M^{-2}+ \\ +4M^{-4}-3M^{-4}d+3M^{-6}]^{1/2}\}^{-1}. \quad (9\,\text{c})$$

Now the last form can be expanded in a series containing only positive powers of $d^{1/2}$ and M^{-2}, as in Chester's solution, only under the assumption that $M^{-2} = O(d^2)$, though Chester assumes merely that $M^{-2} < d$. This minor discrepancy probably indicates a certain lack of fidelity in the present model. To play safe, however, we shall treat Chester's solution only under the additional assumption that $M^{-2} = O(d^2)$. For $\gamma = 7/5$, this restricts M to values greater than perhaps 4 or 5.

As before, the model suggests recasting Chester's series as an approximation for $(1-\eta)^{-1}$. The result, including now terms in ξ^2, is

$$1-\eta = d\left[1+\sqrt{\left(\frac{8d}{3}\right)}-\frac{13}{30}d+\frac{187}{840}d\sqrt{\left(\frac{8d}{3}\right)}-\ldots\right]^{-1}\times$$
$$\times\left[1+\xi^2\left\{\frac{7}{12}+\frac{1}{60}\sqrt{\left(\frac{8d}{3}\right)}-\frac{299}{1008}d+\frac{M^{-2}}{3d}-\right.\right.$$
$$\left.\left.-\frac{27269}{50400}d\sqrt{\left(\frac{8d}{3}\right)}+\ldots\right\}\right]. \quad (10)$$

(Note that on the axis the solution to this degree of approximation does not depend upon M explicitly, but only in so far as it appears in d.) The apparent convergence of the terms in ξ^2, though again not as rapid as that of the terms independent of ξ, is a great improvement over that of the original series, and probably adequate for practical purposes.

For $\gamma = 7/5$, this modified series (10) leads to ratios of stand-off distance Δ, initial shock-wave radius R_s, and body nose radius R_b, which are compared in table 1 with the results of the model.

M		1·25	1·5	2	3	4	5	6	8	10	∞
Δ/R_s	eqn.(10)	·2815	·2405	·1890	·1429	·1242	·1150	·1098	·1046	·1021	·0976
	model	·2443	·2109	·1693	·1323	·1174	·1100	·1059	·1017	·0997	·0962
R_b/R_s	eqn.(10)	·2783	·3608	·4766	·5934	·6450	·6715	·6868	·7025	·7101	·7238
	model	·3080	·3966	·5033	·6014	·6407	·6594	·6695	·6796	·6842	·6923
Δ/R_b	eqn.(10)	1·0112	·6667	·3966	·2408	·1926	·1713	·1599	·1489	·1438	·1349
	model	·7932	·5317	·3364	·2200	·1832	·1668	·1582	·1496	·1458	·1389

Table 1.

For $M = \infty$ an accurate numerical solution of the full equations has been found as the first part of a current programme of calculating the flow field behind a family of detached shock waves. The results are $\Delta/R_s = 0{\cdot}0988$ $R_b/R_s = 0{\cdot}726$, and $\Delta/R_b = 0{\cdot}1360$. Hence the modified series predicts the ratio of stand-off distance to body radius to within a fraction of 1%, and even the model is only 2% off at $M = \infty$. The numerical solution also indicates that the axisymmetric body that supports a parabolic shock wave at $M = \infty$ is itself almost exactly parabolic (whereas in plane flow it is exceedingly close to circular).

The tabulated values of stand-off distance are compared in figure 3 with a number of measurements on spheres in air under conditions such that the adiabatic exponent is close to 7/5. Schlieren photographs show that for spheres flying at Mach numbers between 2 and 7 the detached shock wave is nearly parabolic over the region that determines the stand-off distance, so that the comparison is justified. At lower Mach numbers the shock wave on a sphere departs significantly from a parabola, and the assumptions of Chester's theory have also been greatly exceeded, so that the agreement shown in figure 3 is undoubtedly coincidental. (No values are shown for Chester's original series, since it converges so poorly that he did not himself suggest using it at $\gamma = 7/5$.)

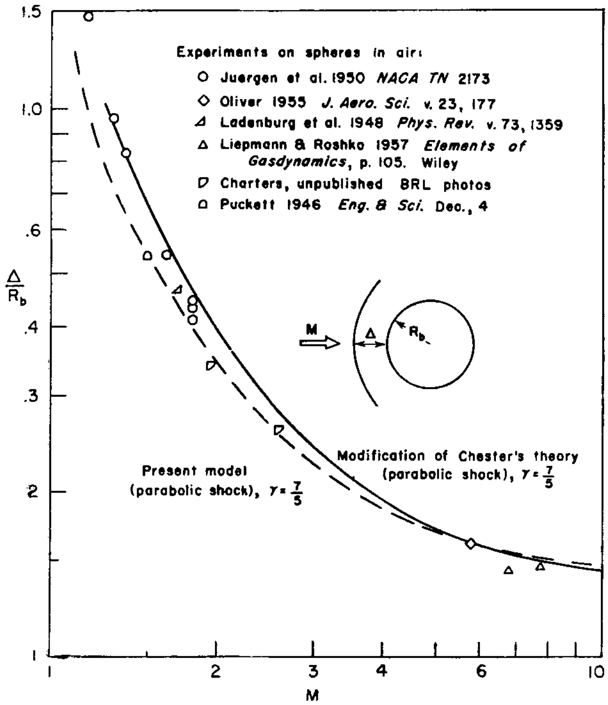

Figure 3. Stand-off distance of shock wave from sphere.

3.3. *Pressure distribution*

Chester's series for the pressure at the surface of the body also suffers from poor convergence (except at the stagnation point). Thus at $M = \infty$ it gives, in units of $\rho_0 V^2$

$$p = \left(1 - \frac{1}{2}\delta + \frac{1}{8}\delta^2\right) - r^2\left[\frac{4}{3} + \frac{253}{63}\delta - \frac{32}{9}\delta\surd(8\delta/3) + \frac{2\,505\,007}{13\,860}\delta^2\right]. \quad (11)$$

The corresponding result for the model stream function is found, using the Bernoulli equation and the condition of constant entropy along streamlines, to be

$$p = \left(1 - \frac{1}{2}\delta + \frac{1}{8}\delta^2\right) - r^2\left[\frac{3}{2} + \frac{27}{4}\delta - 9\delta\surd(9\delta/3) + \frac{747}{16}\delta^2\right]. \quad (12)$$

The agreement is fair. In particular, the model confirms that the last coefficient is enormous, though Chester's is still greater by a factor of four.

Unfortunately, the model does not in this case lead to a useful modification of Chester's solution. It actually suggests that his series should be used to calculate the quantity $\eta^4\rho^2(u^2+v^2)$, and that the pressure then be found from the Bernoulli and entropy relations; but working back from Chester's solution for p gives a series for that quantity whose convergence is not appreciably improved.

In the case of plane flow, the convergence of Chester's series is nearly adequate. It would nevertheless be of interest to construct a model, if only to explain the source of the logarithmic terms. (Successively higher powers of logarithms in perturbation series may be the asymptotic representation of Bessel functions, inverse hyperbolic functions, small fractional powers, etc.). However, the author has been unable to devise a useful model, or otherwise to discover an appropriate transformation of the series.

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